Mechanics of Materials

Chapter 2

Strain
2.1 Introduction

- So far dealt mainly with the strength of structural member. Here we begin our study of an equally important topic of mechanics—deformations, or strains.

- In general terms, strain is a geometric quantity that measures the deformation of a body. There are two types of strain: normal strain, which characterizes dimensional changes, and shear strain, which describes distortion (changes in angles).

- Stress and strain are two fundamental concepts of mechanics of materials. Their relationship to each other defines the mechanical properties of a material, the knowledge of which is of the utmost importance in design.

- Use force-deformation relationships in conjunction with equilibrium analysis to solve statically indeterminate problems.
2.2 Axial Deformation；Stress-Strain Diagram

- The strength of a material is not the only criterion that must be considered when designing machine parts or structures. The stiffness of a material is often equally important properties such as hardness, toughness, and ductility. These properties are determined by laboratory tests.

- Many materials, particularly metals, have established standards that describe the test procedures in detail. We will confine our attention to only one of the tests—the tensile test of steel—and use its results to illustrate several important concepts of material behavior.
a. Normal (axial) strain

- The elongation $\delta$ may be caused by an applied axial force, or an expansion due to an increase in temperature, or even a force and a temperature increase acting simultaneously.

$$\varepsilon = \frac{\Delta x}{x}$$

Figure 2.1 Deformation of a prismatic bar.

- Strain describes the geometry of deformation. The normal strain $\varepsilon$ (lowercase Greek epsilon) is defined as the elongation per unit length. Therefore, the normal strain in the bar in the axial direction, also known as the axial strain, is

$$\varepsilon = \frac{\delta}{L}$$
If the bar deforms uniformly, then Eq. (2.1) expression should be viewed as the *average axial strain*. Note that normal strain, being elongation per unit length, is a *dimensionless quantity*. However, “units” such as in./in. or mm/mm are frequently used for normal strain.

If the deformation is not uniform, we let $O$ be a point in the bar located at the distance $\chi$ from the fixed end. We define the axial strain at point $O$ as

$$
\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta \delta}{\Delta \chi} = \frac{d \delta}{d \chi}
$$

(2.2)
We note that if the distribution of the axial strain $\varepsilon$ is known, the elongation of the can be computed from

$$\delta = \int_{\delta_0}^{\delta_L} d\delta = \int_{\chi_0}^{\chi_L} \varepsilon d\chi$$

(2.3)

For uniform strain distribution, Eq. (2.3) yields $\delta = \varepsilon L$, which agrees with Eq.(2.1).

The results are also applicable to compression. By convention, compression (shortening) carries a negative sign.
b. Tension test

In the standard tension test, the specimen shown in Fig. 2.2 is placed in the grips of a testing machine. The grips are designed so that the load $P$ applied by the machine is axial. Two gage marks are scribed on the specimen to define the gage length $L$.

These marks are loaded away from the ends to avoid the load effects caused by the grips and to ensure that the stress and strain are uniform in the material between the marks. The testing machine elongates the specimen at a slow, constant rate until the specimen ruptures.

Figure 2.2 Specimen used in the standard tension test.
During the test, continuous readings are taken of the applied load and the elongation of the gage length. These data are then converted to stress and strain.

The stress is obtained from $\sigma = P/A$, where $P$ is the load and $A$ represents the original cross-sectional area of the specimen. The strain is computed from $\varepsilon = \sigma / L$, where $\delta$ is the elongation where $\delta$ is the elongation between the gage marks and $L$ is the original gage length. These results are referred to as nominal stress and nominal strain.

As the bar is being stretched, its cross-sectional area is reduced and the length between the gage marks increases. Dividing the load by the actual (current) area of the specimen, we get the true stress. Similarly, the true strain is obtained by dividing the elongation $\delta$ by the current gage length.
The nominal and true measures are essentially the same in the working range of metals. They differ only for very large strains. With only a few exceptions, engineering applications use nominal stress and strain.

A stress-strain diagram for structural steel is shown in Fig. 2.3. The following mechanical properties can be determined from the diagram.

Figure 2.3 Stress-strain diagram obtained from the standard tension test on a structural steel specimen.
**Proportional Limit and Hooke’s Law**

As seen in Fig. 2.3, the stress-strain diagram is a straight line from the origin O to a point called the **proportional limit**. This plot is a manifestation of **Hooke’s law**: Stress is proportional to strain; that is,

\[ \sigma = E \varepsilon \]  

(2.4)

where \( E \) is material property known as the **modulus of elasticity** or **Young’s modulus**. The units of \( E \) are the same as the units of, Pa or psi. For steel, \( E = 29 \times 10^6 \) psi, or 200 GPa, approximately.
Note that *Hooke’s law* does not apply to the entire diagram; its validity ends at the proportional limit. Beyond this point, stress is no longer proportional to strain.

**Elastic Limit** A material is said to be *elastic* if, after being loaded, the material returns to its original shape when the load is removed. The stress beyond which the material is no longer elastic. The permanent deformation that remains after the removal of the load is called the *permanent set*. The elastic limit is slightly larger than the proportional limit.
- **Yield Point**: The point where the stress-strain diagram becomes almost horizontal is called the *yield point*, and the corresponding stress is known as the *yield stress* or *yield strength*.

- Beyond the yield point there is an appreciable elongation of the material without a corresponding increase in load. Indeed, the load may actually decrease while the yielding occurs.
However, the phenomenon of yielding is unique to structural steel. Other grades of steel, steel alloys, and other material do not yield, as indicated by the stress-strain curves of the materials shown in Fig. 2.4.

After repeated loading, these residual stresses are removed and the stress-strain curves become practically straight lines.
For materials that do not have a well-defined yield point, yield stress is determined by the **offset method**. This method consists of drawing a line parallel to the **initial tangent** of the stress-strain curve; this line starts at a prescribed offset strain, usually 0.2% (\( \varepsilon = 0.002 \)). The intersection of this line with the stress-strain curve, is called the **yield point at 0.2% offset**.

**Ultimate Stress** The **ultimate stress** or **ultimate strength**, as it is often called, is the **highest stress on** the stress-strain curve.
- **Rupture Stress** The rupture strength is the stress at which failure occurs. The nominal rupture strength is computed by dividing the load at rupture by the original cross-sectional area. The true rupture strength is calculated using the reduced area of the cross section where the fracture occurred.

- The difference in the two values results from a phenomenon known as *necking*. As failure approaches, the material stretches very rapidly, causing the cross section to narrow, as shown in Fig. 2.6. However, the ultimate strength is commonly used as the maximum stress that the material can carry.

![Failed tensile test specimen showing necking, or narrowing, of the cross section.](image)
C. Working stress and factor of safety

The working stress $\sigma_w$, also called the *allowable stress*, is the maximum safe axial stress used in design. In most design, the working stress should be limited to values **not exceeding the proportional limit** so that the stresses remain in the elastic range. However, because the proportional limit is **difficult to determine accurately**, it is customary to base the working stress on either the yield stress $\sigma_{yp}$ or the ultimate stress $\sigma_{ult}$, divided by a suitable number $N$, called the factor of safety. Thus,

$$\sigma_w = \frac{\sigma_{yp}}{N} \text{ or } \frac{\sigma_{ult}}{N}$$  \hspace{1cm} (2.5)

The yield point is selected as the basis for determining $\sigma_w$ in structural steel because it is the stress at which a prohibitively large permanent set may occur. For other material, the working stress is usually based on the ultimate strength.
usually the working stress is set by a group of experienced engineers and is embodied in building codes and specifications. In many materials the proportional limit is about one-half the ultimate strength.

To avoid accidental overloading, a working stress of one-half the proportional limit is usually specified for dead loads that are gradually applied (The term dead load refers to the weight of the structure and other loads that, once applied are not removed.)

A working stress set in this way corresponds to a factor of safety of 4 with respect to $\sigma_{ult}$ and is recommended for materials that are known to be uniform and homogenous.

For other materials, such as wood, in which unpredictable nonuniformities (such as knotholes) may occur, larger factors of safety are used. The dynamic effect of suddenly applied loads also requires higher factors of safety.
2.3 Axially Loaded Bars

- The stress caused by $P$ is below the proportional limit, so that Hooke’s law $\sigma = E \cdot \varepsilon$ is applicable. Because the bar deforms uniformly, the axial strain is $\varepsilon = (\delta / L)$. Therefore, the elongation of the bar is

$$\delta = \frac{\sigma L}{E} = \frac{PL}{EA}$$  \hspace{1cm} (2.6)

If the strain (or stress) in the bar is not uniform then Eq. (2.6) is invalid.

Figure 2.7 Axially loaded bar.
In the case where the axial strain varies with the x-coordinate, the elongation of the bar can be obtained by integration, as stated in Eq. (2.3) \( \delta = \int_0^L \varepsilon dx \). Using \( \varepsilon = \frac{\sigma}{E} = \frac{P}{EA} \). Where P is the internal axial force, we get

\[
\delta = \int_0^L \frac{\sigma}{E} dx = \int_0^L \frac{P}{EA} dx
\]  

(2.7)

Eq. (2.7) reduces to Eq. (2.6) only if \( P \), \( E \), and \( A \) are constants.
Notes on the Computation of Deformation

- The magnitude of the internal force $P$ in Eqs. (2.6) and (2.7) must be found equilibrium analysis. Note that a positive (tensile) $P$ results in positive $\delta$ (elongation);

\[ \delta = \frac{\sigma L}{E} = \frac{PL}{EA} \]

\[ \delta = \int_0^L \frac{\sigma}{E} \, dx = \int_0^L \frac{P}{EA} \, dx \]

- In the U.S. Customary system, $E$ is expressed in psi ($lb/in.^2$), so that the units of the other variables should be $P$ [lb], $L$ [in.], and $A$ [in.^2]. In the SI system, where $E$ is in Pa (N/m^2), the consistent units are $P$[N], $L$[m], and [m^2].

- As long as the axial stress is in the elastic range, the elongation (or shortening) of a bar is very small compared to its length. This property can be utilized to simplify the computation of displacements in structure containing axially loaded bars, such as trusses.
Sample Problem 2.1

The steel propeller shaft $ABCD$ carries the axial loads shown in Fig. (a). Determine the change in the length of the shaft caused by these loads. Use $E = 29 \times 10^6$ psi for steel.

**Solution**

the internal forces in the three segments of the shaft are

$P_{AB} = P_{BC} = 2000$ lb (T)  
$P_{CD} = -4000$ lb (C)
Noting that tension cause elongation and compression results in shortening we obtain for the elongation of the shaft

\[
\delta = \sum \frac{PL}{EA} = \frac{1}{E} \left[ \left( \frac{PL}{A} \right)_{AB} + \left( \frac{PL}{A} \right)_{BC} - \left( \frac{PL}{A} \right)_{CD} \right]
\]

\[
= \frac{1}{29 \times 10^6} \left[ \frac{2000(5 \times 12)}{\pi (0.5)^2 / 4} + \frac{2000(4 \times 12)}{\pi (0.75)^2 / 4} - \frac{4000(4 \times 12)}{\pi (0.75)^2 / 4} \right]
\]

\[
= \frac{(611.2 + 217.3 - 434.6) \times 10^3}{29 \times 10^6}
\]

\[
= 0.01358 \text{ in. (elongation)} \quad \text{Answer}
\]
Sample Problem 2.2

The cross section of the 10-m-long flat steel bar $AB$ has a constant thickness of 20 mm, but its width varies as shown in the figure. Calculate the elongation of the bar due to the 100 kN axial load. Use $E = 200$ GPa.

Determining $A$ as a function of $x$. The cross-sectional areas at $A$ and $B$ are $A_A = 20 \times 40 = 800$ mm$^2$ and $A_B = 20 \times 120 = 2400$ mm$^2$.

$$A = A_A + \left( A_B - A_A \right) \frac{x}{L} = 800 \text{mm}^2 + \left( 1600 \text{mm}^2 \right) \frac{x}{L}$$
Converting the from mm\(^2\) to m\(^2\) and substituting \(L = 10\) m, we get
\[
A = (800 + 16x) \times 10^{-6} \text{ m}^2 \quad \text{(a)}
\]
Substituting Eq. (a) together with \(P = 100 \times 10^3\) N and \(E = 200 \times 10^9\) Pa into Eq. (2.7), we obtain for the elongation of the rod
\[
delta = \int_0^L \frac{P}{EA} \, dx = \int_0^{10m} \frac{100 \times 10^3}{(200 \times 10^9)[(800 + 160x) \times 10^{-6}]} \, dx
\]
\[
= 0.5 \int_0^{10m} \frac{dx}{800 + 160x} = \frac{0.5}{160} \left[ \ln(800 + 160x) \right]_0^{10}
\]
\[
= \frac{0.5}{160} \ln \frac{2400}{800} = 3.43 \times 10^{-3} \text{ m} = 3.43 \text{ mm} \quad \text{Answer}
Sample Problem 2.3

The rigid bar $BC$ in Fig. (a) is supported by the steel rod $AC$ of cross-sectional area $0.25$ in.$^2$. Find the vertical displacement of point $C$ caused by the 2000-lb load. Use $E=29\times10^6$ psi for steel.

Solution

computing the axial force in rod $AC$. Noting that bar $BC$ is a two-force body, the FBD of joint $C$ in Fig. (b) yields

$$\sum F_y = 0 \quad \uparrow P_{AC} \sin 40^\circ - 2000 = 0$$

$$P_{AC} = 3111\text{lb}$$
Noting that the length of the rod is

\[ L_{AC} = \frac{L_{BC}}{\cos 40^\circ} = \frac{8 \times 12}{\cos 40^\circ} = 125.32 \text{ in.} \]

\[ \delta_{AC} = \left( \frac{PL}{EA} \right)_{AC} = \frac{3111(125.32)}{(29 \times 10^6)(0.25)} = 0.05378 \text{ in.} \quad \text{(elongation)} \]

\[ \Delta_C = \frac{\delta_{AC}}{\sin 40^\circ} = \frac{0.05378}{\sin 40^\circ} = 0.0837 \text{ in.} \quad \downarrow \]

Answer
2.4 **Generalized Hook’s Law**

**a. Uniaxial loading; poisson’s ratio**

As illustrated in Fig. 2.8. In 1811, Poisson showed that the ratio of the transverse strain to the axial strain is constant for stresses within the proportional limit. This constant, called *poisson’s ratio*, is denoted by $\nu$ (lowercase Greek $\nu$).

For uniaxial loading in the x-direction, as in Fig. 2.8, Poisson’s ratio is $\nu = -\frac{\varepsilon_t}{\varepsilon_x}$, where $\varepsilon_t$ is the transverse strain.

**Figure 2.8** Transverse dimensions contract as the bar is stretched by an Axial force $P$. 
The minus sign indicates that a positive strain (elongation) in the axial direction causes a negative strain (contraction) in the transverse directions.

The transverse strain $\varepsilon_t$ is uniform throughout the cross section and is the same in any direction in the plane of the cross section. Therefore, we have for uniaxial loading

$$\varepsilon_y = \varepsilon_z = -\nu \varepsilon_x$$

(2.8)

Poisson’s ratio $\nu$ is a dimensionless quantity that ranges between 0.25 and 0.33 for metals.

Using $\sigma_x = E \varepsilon_x$ in Eq.(2.8) yields the generalized Hook’s law for uniaxial loading ($\sigma_y = \sigma_z = 0$):

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \varepsilon_y = \varepsilon_z = -\nu \frac{\sigma_x}{E}$$

(2.9)
b. Multiaxial loading

**Biaxial Loading** Poisson’s ratio permits us to extend Hooke’s law for uniaxial loading to biaxial and triaxial loadings. Consider in Fig. 2.9(a). The strains caused by $\sigma_\chi$ along are given in Eqs. (2.9). Similarly, the strains due to $\sigma_y$ are $\varepsilon_y = \sigma_y / E$ and $\varepsilon_\chi = \varepsilon_z = -\nu \sigma_y / E$.

![Figure 2.9](image.png)

**Figure 2.9** (a) Stresses acting on a material element in biaxial loading.
Using superposition, we write the combined effect of the two normal stresses as

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) \\
\varepsilon_y &= \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) \\
\varepsilon_z &= -\frac{\nu}{E} \left( \sigma_x + \sigma_y \right)
\end{align*}
\] (2.10)

which is Hooke’s law for biaxial loading in the \(xy\)-plane \( (\sigma_z = 0) \).

The first two of Eqs. (2.10) can be inverted to express the stresses in terms of the strains:

\[
\begin{align*}
\sigma_x &= \frac{(\varepsilon_x + \nu \varepsilon_y)E}{1 - \nu^2} \\
\sigma_y &= \frac{(\varepsilon_y + \nu \varepsilon_x)E}{1 - \nu^2}
\end{align*}
\] (2.11)
Two-dimensional views of the stresses and the resulting deformation in the $xy$-plane are shown in Figs. 2.9(b) and (c). Note that Eqs.(2.10) show that for biaxial loading $\varepsilon_z$ is not zero; that is, the strain is triaxial rather than biaxial.

$$
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) \\
\varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) \\
\varepsilon_z = -\frac{\nu}{E} \left( \sigma_x + \sigma_y \right)
$$

(2.10)
**Triaxial Loading**  Hooke’s law for the triaxial loading in Fig. 2.10 is obtained by adding the contribution of $\sigma_z$, $\varepsilon_z = \frac{\sigma_z}{E}$ and $\varepsilon_x = \varepsilon_y = -\nu \frac{\sigma_z}{E}$, to the strains in Eqs. (2.10), which yields

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \\
\varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \nu (\sigma_z + \sigma_x) \right] \\
\varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right]
\end{align*}
\]  (2.12)

**Figure 2.10**  Stresses acting on a material element in triaxial loading.

Equations (2.8)-(2.12) are valid for both tensile and compressive effects. It is only necessary to assign positive signs to elongations and tensile stresses.
c. **Shear loading**

- Shear stress causes the deformation shown in Fig. 2.11. The lengths of the sides of the element do not change, but the element undergoes a distortion from a rectangle to a parallelogram. The *shear strain*, which measures the amount of distortion, is the angle $\gamma$ (lowercase Greek *gamma*), always expressed in radians.

![Image of shear deformation](image)

**Figure 2.11** Deformation of a material element caused by shear stress.
It can be shown that the relationship between shear stress $\tau$ and shear strain $\gamma$ is linear within the elastic range; that is,

$$\tau = G \gamma$$  \hspace{1cm} (2.13)

Which is Hooke's law for shear. The material constant $G$ is called the shear modulus of elasticity (or simply shear modulus), or the modulus of rigidity. The shear modulus has the same units as the modulus of elasticity (Pa or psi).

The shear modulus of elasticity $G$ is related to the modulus of elasticity $E$ and Poisson's ratio $\nu$ by

$$G = \frac{E}{2(1 + \nu)}$$  \hspace{1cm} (2.14)
Sample problem 2.4

The 50-mm-diameter rubber rod is placed in a hole with rigid, lubricated walls. There is no clearance between the rod and the sides of the hole. Determine the change in the length of the rod when the 8-kN load is applied. Use $E = 40 \text{ MPa}$ and $\nu = 0.45$ for rubber.

Solution

Lubrication allows the rod to contract freely in the axial direction, so that the axial stress throughout the bar is

$$\sigma_x = \frac{P}{A} = -\frac{8000}{\frac{\pi}{4}(0.05)^2} = -4.074 \times 10^6 \text{ pa}$$
Because the walls of the hole prevent transverse strain in the rod, we have $\varepsilon_y = \varepsilon_z = 0$. The tendency of the rubber to expand laterally (Poisson’s effect) is resisted by the uniform contact pressure $p$ between the walls and rod, so that $\sigma_y = \sigma_z = -p$. If we use the second of Eqs. (2.12), the condition $\varepsilon_y = 0$ becomes

$$\frac{\sigma_y - \nu (\sigma_z + \sigma_x)}{E} = -\frac{p - \nu (-p + \sigma_x)}{E} = 0$$

$$p = \frac{\nu \sigma_x}{1 - \nu} = \frac{0.45(-4.074 \times 10^6)}{1 - 0.45} = 3.333 \times 10^6 \text{ Pa}$$

The axial strain $\varepsilon_x$ is given by the first of Eqs. (2.12):

$$\varepsilon_x = \frac{\sigma_x - \nu (\sigma_y + \sigma_z)}{E} = \frac{\sigma_x - \nu (-2p)}{E}$$

$$= \frac{[-4.074 - 0.45(-2 \times 3.333)] \times 10^6}{40 \times 10^6} = -0.02686$$
The corresponding change in the length of the rod is

\[ \delta = \varepsilon_x L = -0.0268 \times 300 \]

\[ = -8.06 \text{ mm} = 8.06 \text{ mm} \quad (\text{contraction}) \quad \text{Answer} \]

For comparison, note that if the constraining effect of the hole were neglected, the deformation would be

\[ \delta = \frac{PL}{EA} = \frac{8000(0.3)}{(40 \times 10^6) \left[ \frac{\pi}{4} (0.05)^2 \right]} \]

\[ = -0.0306m = -30.6\text{mm} \]
**Sample Problem 2.5**

Two 1.75-in. thick rubber pads to form the shear mount shown in Fig. (a). Find the displacement of the middle plate when the 1200-lb load is applied. Consider the deformation of rubber only. Use \( E = 500 \text{ psi} \) and \( \nu = 0.48 \).

![Figure (a) and (b)](image)

**Solution**

When the load is applied, the grid deform as for small regions at the edges of the pad (Saint Venant’s principle).
Each rubber pad has a shear area of $A = 5 \times 9 = 45 \text{ in.}^2$ that carries half the 1200-lb load. Hence, the average shear in the rubber is

$$\tau = \frac{V}{A} = \frac{600}{45} = 13.333 \text{ psi}$$

In Fig. (c), the corresponding shear strain is $\gamma = \frac{\tau}{G}$, where from Eq. (2.14),

$$G = \frac{E}{2(1+\nu)} = \frac{500}{2(1+0.48)} = 168.92 \text{ psi}$$

$$\gamma = \frac{\tau}{G} = \frac{13.333}{168.92} = 0.07893$$

From Fig. (b), the displacement of the middle plate is

$$t\gamma = 1.75(0.07893) = 0.1381 \text{ in.} \quad \text{Answer}$$
2.5 Statically Indeterminate Problems

- If the equilibrium equations are sufficient to calculate all the forces (including support reactions) that act on a body, these forces are said to be *statically determinate*.

- In *statically determinate problems*, the number of unknown forces is always equal to the number of independent equilibrium equations.

- If the number of unknown forces exceeds the number of independent equilibrium equations, the problem is said to be *statically indeterminate*.

- A statically indeterminate problem always has geometric restrictions imposed on its deformation. The mathematical expressions of these restrictions known as the *compatibility equations*, provide us with the additional equations needed to solve the problem.
Because the source of the compatibility equations is deformation, these equations contain as unknowns either strains or elongations. Use Hooke’s law to express the deformation measures in terms of stresses or forces. The equations of equilibrium and compatibility can then be solved for the unknown forces (force-displacement equation).

**Procedure for Solving Statically Indeterminate Problem**

- Draw the required free-body diagrams and derive the equations of equilibrium.
- Derive the compatibility equations. To visualize the restrictions on deformation, it is often helpful to draw a sketch that exaggerates the magnitudes of the deformations.
- Use Hooke’s law to express the deformations (strains) in the compatibility equations in terms of forces (or stresses)
- Solve the equilibrium and compatibility equations for the unknown forces (force-displacement equation).
Sample Problem 2.6

The concrete post in Fig. (a) is reinforced axially with four symmetrically placed steel bars, each of cross-sectional area 900 mm$^2$. Compute the stress in each material when the 1000-kN axial load is applied. The moduli of elasticity are 200 Gpa for steel and 14 Gpa for concrete.
Compatibility

The changes in lengths of the steel rods and concrete must be equal; that is, \( \delta_{st} = \delta_{co} \), the compatibility equation, written in term of strains, is \( \epsilon_{st} = \epsilon_{co} \) (b)

**Hooke’s law** (force-displacement equation) From Hooke’s law, Eq. (b) becomes

\[
\delta = \frac{\sigma L}{E} = \frac{PL}{EA} \quad \frac{\sigma_{st}}{E_{co}} = \frac{\sigma_{co}}{E_{co}}
\] (c)

Solution

\[
\sum F = 0 \pm \uparrow \quad 4P_{st} + P_{co} - 1.0 \times 10^6 = 0
\]

Equilibrium

\[
\sum F = 0 \pm \uparrow \quad P_{st} + P_{co} - 1.0 \times 10^6 = 0
\]

\[
\sigma_{st} A_{st} + \sigma_{co} A_{co} = 1.0 \times 10^6 N \quad (a)
\]

the problem is statically indeterminate.
Equations (a) and (c) can now be solved for the stresses. From Eq. (c) we obtain

\[ \sigma_{st} = \frac{E_{st}}{E_{co}} \sigma_{co} = \frac{200}{14} \sigma_{co} = 14.286 \sigma_{co} \quad \text{(d)} \]

Substituting the cross-sectional areas

\[ A_{st} = 4 \times (900 \times 10^{-6}) = 3.6 \times 10^{-3} \text{ m}^2 \]
\[ A_{co} = 0.3^2 - 3.6 \times 10^{-3} = 86.4 \times 10^{-3} \text{ m}^2 \]

and Eq. (d) into Eq. (a) yields

\[ (14.286 \sigma_{co})(3.6 \times 10^{-3}) + \sigma_{co}(8.64 \times 10^{-3}) = 1.0 \times 10^6 \]

Solving for the stress in concrete, we get

\[ \sigma_{co} = 7.255 \times 10^6 \text{ Pa} = 7.255 \text{Mpa} \quad \text{Answer} \]

From Eq. (d), the stress in steel is

\[ \sigma_{st} = 14.286 \times (7.255) = 103.6 \text{ MPa} \quad \text{Answer} \]
**Sample Problem 2.7**

Let the allowable stresses in the post described in Sample Problem 2.6 be $\sigma_{st} = 120$ Mpa and $\sigma_{co} = 6$ Mpa. Compute the maximum safe axial load $P$ and may be applied.

**Solution**

substituting the allowable stresses into the equilibrium equation—see Eq. (a) in Sample Problem 2.6. This approach is **incorrect** because it ignores the compatibility condition, the equal strains of the two materials, $\delta_{st} = \delta_{co}$, From Eq. (d) in Sample problem 2.6,

$$\sigma_{st} = 14.286 \sigma_{co} \text{ (the concrete were broked rather than steel)}$$

Therefore, if the concrete were stressed to its limit of 6 Mpa. The corresponding stress in the steel would be

$$\sigma_{st} = 14.286(6) = 85.72 \text{ MPa}$$

which is below the allowable stress of 120 MPa.
The maximum safe axial load is thus found by substituting $\sigma_{co} = 6$ MPa and $\sigma_{st} = 85.72$ MPa (rather than $\sigma_{st} = 120$ MPa) into the equilibrium equation:

$$P = \sigma_{st}A_{st} + \sigma_{co}A_{co}$$

$$= (85.72 \times 10^6) (3.6 \times 10^{-3}) + (6 \times 10^6) (86.4 \times 10^{-3})$$

$$= 827 \times 10^3 \text{ N} = 827 \text{ kN}$$

*The maximum safe axial load* 之計算判斷式

由 Compatibility 条件，配合 force-displacement equation $\delta = PL/AE$ 转成 $\sigma L/E$，代入比较判断何者之 the allowable stresses 是控制破坏之允许应力与其对应的另外之允许应力，再计算

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$
Sample Problem 2.8

Figure (a) shows a copper rod that is placed in an aluminum sleeve. The rod is 0.005 in. longer than the sleeve. Find the maximum safe load $P$ that can be applied to the bearing plate, using the following data:

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<tr>
<th></th>
<th>Copper</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (in.$^2$)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$E$ (psi)</td>
<td>$17\times10^6$</td>
<td>$10\times10^6$</td>
</tr>
<tr>
<td>Allowable stress (ksi)</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>
**Solution**

*Equilibrium* in Fig. (b). From this FBD we get

$$\sum F = 0 + \uparrow \ P_{\text{cu}} + P_{\text{al}} - P = 0 \quad (a)$$

Because no other equations of equilibrium are available, the forces $P_{\text{cu}}$ and $P_{\text{al}}$ are statically indeterminate.

*Compatibility* Figure (c) shows the changes in the lengths of the two material, the compatibility equation is

$$\delta_{\text{cu}} = \delta_{\text{al}} + 0.005 \text{ in.} \quad (b)$$
Hooke’s law  Substituting \( \delta = \frac{PL}{AE} = \frac{\sigma L}{E} \) into Eq. (b), we get
\[
\left( \frac{\sigma L}{E} \right)_{cu} = \left( \frac{\sigma L}{E} \right)_{al} + 0.005 \text{ in.} \quad \text{or} \quad \frac{\sigma_{cu}(10.005)}{17 \times 10^6} = \frac{\sigma_{al}(10)}{10 \times 10^6} + 0.005
\]
which reduces to
\[
\delta = \frac{\sigma L}{E} = \frac{PL}{EA} \quad \implies \quad \sigma_{cu} = 1.6992 \sigma_{al} + 8496 \quad \text{(c)}
\]
From Eq. (c), we find that if \( \sigma_{al} = 10000 \text{ psi} \), the copper will be overstressed to 25500 psi. (>\( \sigma_{cu} = 20000 \text{psi} \)). Therefore, the allowable stress \( \sigma_{cu} \) in the copper (20000psi) is the limiting condition. The corresponding stress in the aluminum
\[
20000 = 1.6992 \sigma_{al} + 8496 \quad , \quad \sigma_{al} = 6770 \text{ psi}
\]
From Eq. (a), the safe load is
\[
P = P_{cu} + P_{al} = \sigma_{cu} A_{cu} + \sigma_{al} A_{al}
\]
\[
=20000(2) + 6770(3) = 60300 \text{ lb} = 60.3 \text{ kips}
\]
Answer
**Sample problem 2.9**

Figure (a) shows a rigid bar that is supported by a pin at A and two rods, one made of steel and the other of bronze. Neglecting the weight of the bar, compute the stress in each rod caused by the 50-kN load, the following data:

<table>
<thead>
<tr>
<th></th>
<th>Steel</th>
<th>Bronze</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (mm²)</td>
<td>600</td>
<td>300</td>
</tr>
<tr>
<td>$E$ (GPa)</td>
<td>200</td>
<td>83</td>
</tr>
</tbody>
</table>

(a)
**Solution**

*Equilibrium* in Fig. (b), contains four unknown forces. Since there are only three independent equilibrium equations, these forces are statically indeterminate. The equilibrium equation that does not involve the pin reactions at A is

\[ \sum M_A = 0 + 0.6P_{st} + 1.6P_{br} - 2.4(50 \times 10^3) = 0 \]  \hspace{1cm} (a)

*Compatibility* The displacement of the bar, consisting of a rigid-body rotation about A, is shown greatly exaggerated in Fig. (c).
From similar triangles, we see that the elongations of the supporting rods must satisfy the compatibility condition

\[
\frac{\delta_{st}}{0.6} = \frac{\sigma_{br}}{1.6} \tag{b}
\]

**Hooke’s law** When we substitute \( \delta = \frac{PL}{EA} \) into Eq. (b), the compatibility equation becomes

\[
\frac{1}{0.6} \left( \frac{PL}{EA} \right)_{st} = \frac{1}{1.6} \left( \frac{PL}{EA} \right)_{br}
\]

Using the given data, we obtain

\[
\frac{1}{0.6} \left( \frac{P_{st}}{200} \right) \left( \frac{1.0}{600} \right) = \frac{1}{1.6} \left( \frac{P_{br}}{83} \right) \left( \frac{2}{300} \right)
\]

\[
P_{st} = 3.614P_{br} \tag{c}
\]
Note that we did not convert the areas from mm\(^2\) to m\(^2\), and we omitted the factor 10\(^9\) from the moduli of elasticity. Since these conversion factors appear on both sides of the equation, they would cancel out.

Solving Eqs. (a) and (c), we obtain

\[
P_{st} = 115.08 \times 10^3 \text{ N} \quad P_{br} = 31.84 \times 10^3 \text{ N}
\]

The stresses are

\[
\sigma_{st} = \frac{P_{st}}{A_{st}} = \frac{115.08 \times 10^3}{600 \times 10^{-6}} = 191.8 \times 10^6 \text{ Pa} = 191.8 \text{ MPa} \quad \text{Answer}
\]

\[
\sigma_{st} = \frac{P_{br}}{A_{br}} = \frac{31.84 \times 10^3}{300 \times 10^{-6}} = 106.1 \times 10^6 \text{ Pa} = 106.1 \text{ MPa} \quad \text{Answer}
\]
2.6 Thermal Stresses

- An increase in temperature results in expansion, whereas a temperature decrease produces contraction. This deformation is isotropic (the same in every direction) and proportional to the temperature change.

- It follows that the associated strain, called thermal strain, is

\[ \varepsilon_T = \alpha (\Delta T) \quad (2.15) \]

Where the constant \( \alpha \) is a material property known as the coefficient of thermal expansion, and \( \Delta T \) is the temperature change. The coefficient of thermal expansion represents the normal strain caused by a one-degree change in temperature.

- \( \Delta T \) is taken to be positive when the temperature increases, and negative when the temperature decreases. Thus, in Eq. (2.15), positive \( \Delta T \) produces positive strain (elongation) and negative \( \Delta T \) produces negative strain (contraction).
The units of \( \alpha \) are \( 1/\degree{C} \) (per degree Celsius) in the SI system, and \( 1/\degree{F} \) (per degree Fahrenheit) in the U.S. Customary system. Typical values of \( \alpha \) are \( 23 \times 10^6 /\degree{C} \) (\( 13 \times 10^6 /\degree{F} \)) for aluminum and \( 12 \times 10^6 /\degree{C} \) (\( 6.5 \times 10^6 /\degree{F} \)) for steel.

If the temperature change is uniform throughout the body, the thermal strain \( \varepsilon_T \) is also uniform. Consequently, the change \( \delta_T \) in any dimension \( L \) of the body is given by

\[
\delta_T = \varepsilon_T L = \alpha (\Delta T) L
\]

(2.16)

If thermal deformation is permitted to occur freely (by using expansion joints or roller supports), no internal forces will be induced in the body — there will be strain, but no stress.

In cases where the deformation of a body is restricted, either totally or partially, internal forces will develop that oppose the thermal expansion or contraction. The stresses caused by these internal forces are known as thermal stresses.
The forces that result from temperature changes cannot be determined by equilibrium analysis alone; that is, these forces are statically indeterminate.

The analysis of thermal stresses follows the same principles that we used in Art. 2.5: equilibrium, compatibility, and Hooke’s law.

The only difference here is that we must now include thermal expansion in the analysis of deformation.

\[ \delta_T = \varepsilon_T L = \alpha (\Delta T) L \]
**Procedure for Deriving Compatibility Equations**

- **Remove all constraints** from the body so that the thermal deformation can **occur freely** (this procedure is sometimes referred to as “relaxing the supports”). **Show the thermal deformation** on a sketch using an exaggerated scale.

- **Apply the forces** that are necessary to restore the specified conditions of constraint. **Add the deformations caused by these forces** to the sketch that was drawn in the previous step. (Draw the magnitudes of the deformations so that they are compatible with the geometric constraints.)

- By inspection of the sketch, write **the relationships between the thermal deformations**. And the deformations due to the constraint forces.
Sample problem 2.10

The horizontal steel rod, 2.5 m long and 1200 mm$^2$ in cross-sectional area, is secured between two walls as shown in Fig. (a). If the rod is stress-free at 20 °C, compute the stress when the temperature has dropped to -20°C. Assume that (1) the walls do not move and (2) the walls move together a distance $\triangle = 0.5$ mm. Use $\alpha = 11.7 \times 10^{-6} / ^\circ C$ and $E = 200$ GPa.

Solution

Part 1

Compatibility $\delta_T = \delta_P$

Hooke’s law $\delta_T = \alpha (\triangle T)L$ and $\delta_P = PL/(EA) = \sigma L/E$,

$$\frac{\sigma L}{E} = \alpha (\triangle T)L$$

$\sigma = \alpha (\triangle T) E = (11.7 \times 10^{-6})(40)$

$(200 \times 10^9) = 93.6 \times 10^6$ Pa = 93.6 MPa

Answer
The preceding equation indicates that the stress is independent of the length $L$ of the rod. 

$$\frac{\sigma L}{E} = \alpha(\Delta T)L$$

Part 2

**Compatibility** when the walls move together a distance $\Delta$, 

$$\delta_T = \delta_P + \Delta$$

**Hooke's law** Substituting for $\delta_T$ and $\delta_P$ as in Part 1, we obtain

$$\alpha(\Delta T)L = \frac{\sigma L}{E} + \Delta$$

the stress $\sigma = E \left[ \alpha (\Delta T) - \frac{\Delta}{L} \right] = (200 \times 10^9) \left[ (11.7 \times 10^{-6})(40) - \frac{0.5 \times 10^{-3}}{2.5} \right]$

$$= 53.6 \times 10^6 \text{ Pa} = 53.6 \text{ MPa}$$

the movement of the walls reduces the stress considerably.
Sample Problem 2.11

Figure (a) shows a homogeneous, rigid block weighing 12 kips that is supported by three symmetrically placed rods. The lower ends of the rods were at the same level before the block was attached. Determine the stress in each rod after the block is attached and the temperature of all bars increases by 100 °F. Use the following data:

<table>
<thead>
<tr>
<th></th>
<th>$A$ (in.$^2$)</th>
<th>$E$ (psi)</th>
<th>$\alpha$ (°F$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Each steel rod</td>
<td>0.75</td>
<td>$29 \times 10^6$</td>
<td>$6.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>Bronze rod</td>
<td>1.50</td>
<td>$12 \times 10^6$</td>
<td>$10.0 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
Solution

Compatibility

\[ (\delta_T)_{st} + (\delta_P)_{st} = (\delta_T)_{br} + (\delta_P)_{br} \]

Hooke's law

\[
\left[ \alpha(\Delta T)L \right]_{st} + \left[ \frac{PL}{EA} \right]_{st} = \left[ \alpha(\Delta T)L \right]_{br} + \left( \frac{PL}{EA} \right)_{br}
\]

\[
(6.5 \times 10^{-6}) (100)(2 \times 12) + \frac{P_{st} (2 \times 12)}{(12 \times 10^6)(0.75)}
\]

\[
= (10.0 \times 10^{-6}) (100)(3 \times 12) + \frac{P_{br} (3 \times 12)}{(12 \times 10^6)(1.50)}
\]

0.09195 \[ P_{st} \] − 0.1667 \[ P_{br} \] = 1700 \[ a \]
**Equilibrium**

\[ \Sigma F = 0 + \uparrow 2P_{st} + P_{br} - 12000 = 0 \]  \hspace{1cm} \text{(b)}

Solving Eqs. (a) and (b) simultaneously yields

\[ P_{st} = 8700 \text{ lb and } P_{br} = -5400 \text{ lb} \]

The stresses in the rods are:

\[ \sigma_{st} = \frac{P_{st}}{A_{st}} = \frac{8700}{0.75} 11600 \text{ psi}(T) \]

\[ \text{Answer} \]

\[ \sigma_{br} = \frac{P_{br}}{A_{br}} = \frac{-5400}{1.50} = -3600 \text{ psi} = 3600 \text{ psi}(C) \]

\[ \text{Answer} \]
Sample problem 2.12
Using the data in Sample problem 2.11, determine the temperature increase that would cause the entire weight of the block to be carried by the steel rods.

Solution

Equilibrium  The problem statement implies that the bronze rod is stress-free. Thus, each steel rod carries half the weight of the rigid block, so that $P_{st} = 6000$ lb.

Compatibility  The temperature increase causes the elongations $(\delta_T)_{st}$ and $(\delta_T)_{br}$ in the steel and bronze rods, respectively, as shown in the figure.
Because the bronze rod is to carry no load, the ends of the steel rods must be at the same level as the end of the unstressed bronze rod before the rigid block can be reattached. Therefore, the steel rods must elongate by \((\delta_P)_\text{st}\) due to the tensile forces \(P_\text{st} = 6000\) lb, which gives

\[
(\delta_T)_\text{br} = (\delta_T)_\text{st} + (\delta_P)_\text{st}
\]

**Hooke’s law**

\[
[\alpha(\Delta T)L]_\text{br} = [\alpha(\Delta T)L_\text{st}] + \left[\frac{PL}{EA}\right]_\text{st}
\]

\[
(10 \times 10^{-6})(\Delta T)(3 \times 12) = (6.5 \times 10^{-6})(\Delta T)(2 \times 12) + \frac{6000(2 \times 12)}{(29 \times 10^6)(0.75)}
\]

\[
\Delta T = 32.5 \ ^\circ\text{F} \quad \text{Answer}
\]

As the temperature increase at which the bronze rod would be unstressed. As below the temperature which the bronze rod would be compressed.